An inquiry-based approach to teaching computational thinking in university mathematics

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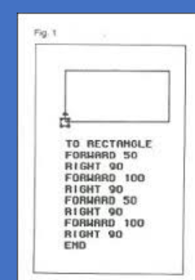
Computational Thinking (CT) in Schools

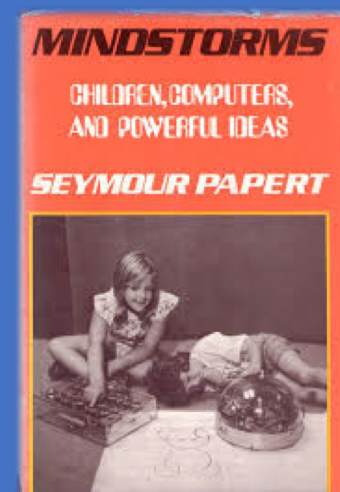
COMPUTATIONAL THINKING

https://educators.brainpop.com/

- Increasing integration of **CT and coding in school curricula** in the last decade; *e.g. in England, New-Zealand, France, Japan, South Africa, Finland, Ontario (Canada), etc.*
- Since 2021**, in PISA's mathematics framework**:
  "students should possess and be able to demonstrate [CT] skills as they apply to mathematics as part of their problem-solving practice."
- "To reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability." (Wing, 2006)
  → CT: Thinking like a computer scientist
- Three key aspects of **CT in math education** (Kallia et al., 2021):
  1. *problem solving* as a fundamental goal of mathematics education in which CT is embedded;
  2. *thinking (cognitive) processes* that include (but are not limited to) abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical and analytical thinking, generalization and evaluation of solutions and strategies; and
  3. phrasing the solution of a mathematical problem in such a way that it can be transferred or outsourced to another person or a machine (*transposition*). (p. 179, *our emphasis*)

- Pioneer Seymour Papert (e.g. 1980),
  → « object-to-think-with »





Broley, Buteau, Modeste, Rafalska, & Stephens (submitted). In *Handbook of Digital Resources in Math Education*

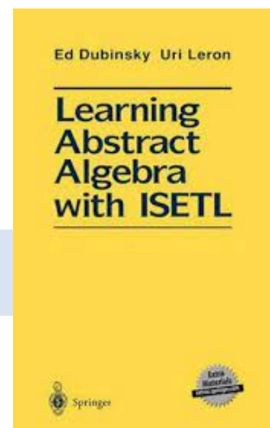
CT in University Mathematics

Integration of programming/computing/CT in our courses and programs:

- As a required skill (Computer Science course requirement)
- Within existing courses; e.g. modeling, numerical analysis
  - E.g., a survey of 46 mathematics departments in the U.K. found that 89% of undergraduate mathematics programs teach programming to all students, most commonly in numerical analysis or statistics (Sangwin & O'Toole, 2017).
- Through a more integrated approach (e.g., throughout a program), as a learning and/or problem-solving tool, e.g.
  - Manchester Metropolitan University in the U.K. (Lynch, 2020)
  - University of Oslo in Norway (Malthe- Sørenssen et al., 2015)
  - Carroll College in the United States (Cline et al., 2020)
  - Brock University in Canada (with an inquiry-based approach)
  - …

Research has shown that programming for university may support students' learning in many areas, such as:

- algebra (Leron & Dubinsky, 1995)
- calculus (Clements, 2020)
- probability (Wilensky, 1995)
- statistics (Mascaró et al., 2016)
- combinatorics (Lockwood & De Chenne, 2020)
- …



Buteau, Broley, Dreise, & Muller (in press). In *EpiDEMES*

In This Seminar Talk:

MICA I-II-III course sequence at Brock University (Canada)

1. Student projects
2. Student engagement
3. Course design and content
4. Teaching approach
5. Evaluation
6. Students' learning experience
7. Implementation constraints and conditions

Mathematics Integrated with Computers and Applications

CTmathU www.ctuniversitymath.ca

Computational Thinking in University Mathematics

Ongoing Research (2017-23)

1. **How do students come to appropriate** 'progmatics', i.e., **programming as an instrument** for authentic pure or applied mathematics investigations?
2. Is this **appropriation sustained over time** (i.e., past course requirement)?
3. **How do instructors create a learning environment** that supports students' instrumental geneses development?

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Canada

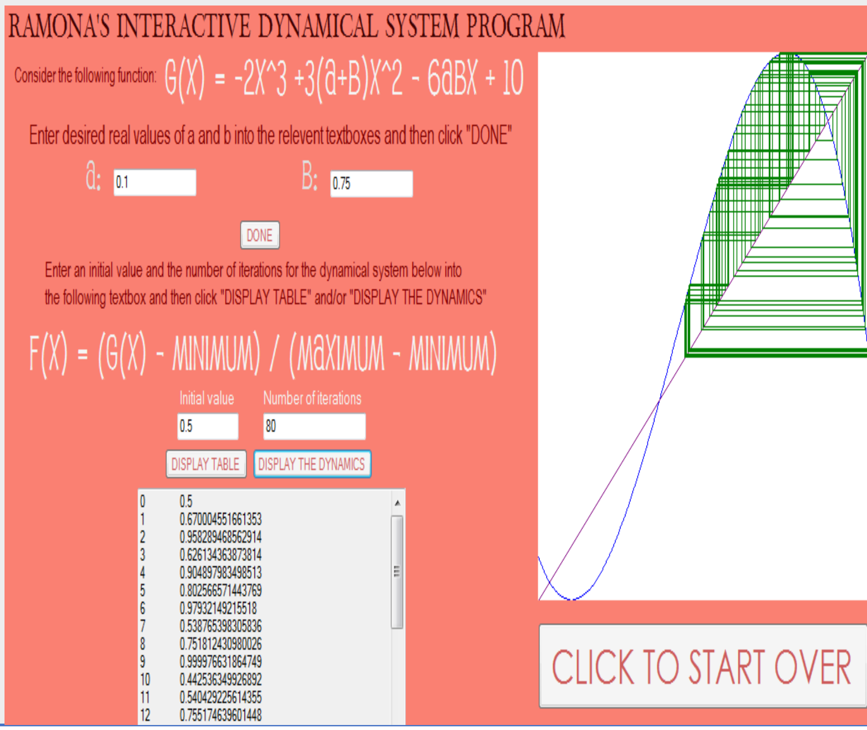
- Case study context: MICA courses
- Naturalistic (i.e. non-interventional), Iterative Design, Mixed-Method
- Longitudinal: following student participants over MICA I-II-III

Buteau, Muller & Ralph (2015). In *Online Proceedings of Coding+Math Symposium.*

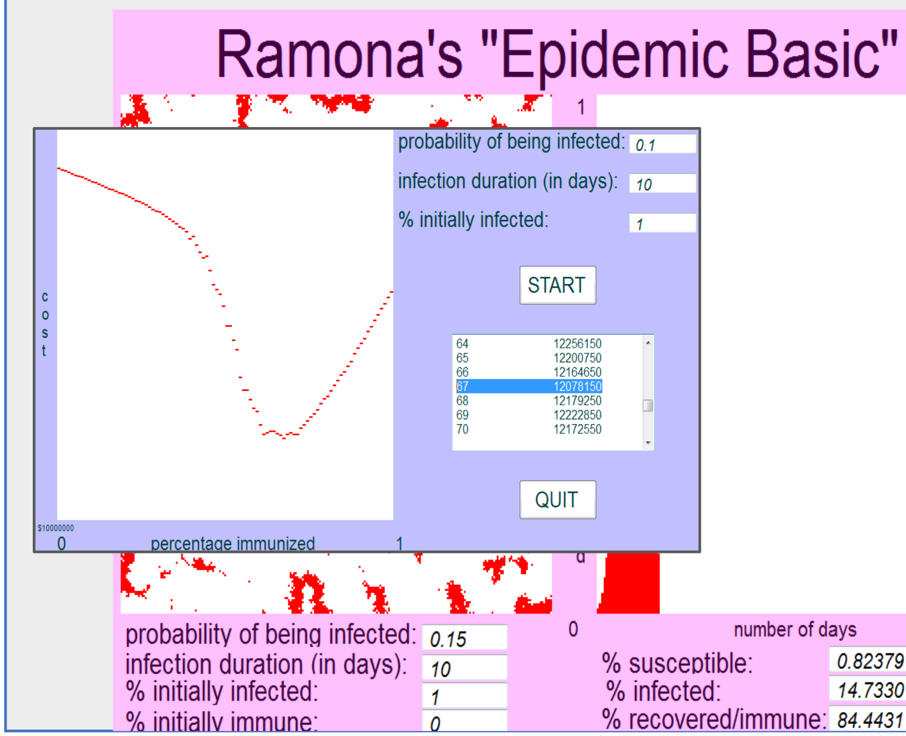
Mathematics Integrated with Computers and Applications

Using Programming to Conduct Pure or Applied Mathematical Inquiries

Students program to explore the dynamics of the dynamical system of the 2-parameter cubics on [0, 1] – MICA I



Students program to simulate the spread of an epidemic using a cellular automaton model – MICA III



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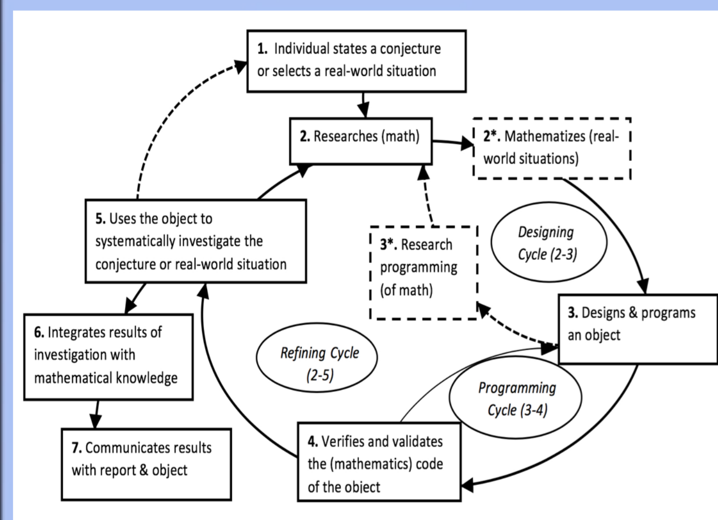
https://ctuniversitymath.ca/research/

Student Engagement Process

Using Programming for Pure/Applied Mathematics Investigation

Development Process Model



Student engagment in MICA projects

1. Formulating a problem/ conjecture to investigate
2. Programming an interactive environment enabling the inquiry
3. Conducting the inquiry using the environment
4. Communicating results

Research Processus from mathematicians who use programming as a tool in their work (Broley, 2015)

Maths pures

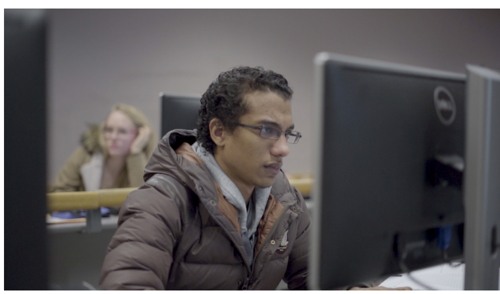
1) choix et délimitation d'un problème mathématique ;
2) création de programmes informatiques ;
   a) développement d'un algorithme ;
   b) codage de l'algorithme ;
   c) vérification et validation du programme ;
3) exploration
   a) observation d'objets mathématiques ;
   b) formulation des conjectures ;
   c) vérification des conjectures ;
4) preuve ;
5) communication des résultats et préparation pour le futur.

(p.56)

Maths appliquées

1) choix d'un phénomène réel ;
2) développement d'un modèle ;
   a) étude du phénomène et des modèles existants ;
   b) identification des aspects importants ;
   c) analyse de données réelles ;
   d) création du modèle mathématique ;
   e) validation analytique du modèle ;
   f) calcul des paramètres ;
3) programmation du modèle ;
   a) recherche sur les outils informatiques existants ;
   b) développement d'un algorithme ;
   c) codage de l'algorithme ;
   d) vérification et validation du programme ;
4) recherche des solutions et expérimentation avec le modèle ;
5) interprétation et validation des résultats ;
6) communication des résultats et préparation pour le futur.

(p.37)



Similar engagement to that of research mathematicians

(Broley, Buteau, & Muller, 2017;

(Buteau, Gueudet, Muller, Mgombelo, & Sacristán, 2019)

Implication: *Having students engage in such projects provides opportunities for students to develop skills needed in professional practice*

Mathematics Integrated with Computers and Applications

MICA courses at Brock University

- **Designed** in 2000 **by practitioner mathematicians**
- **Course Format**: 2 hours lecture + 2 hours lab, weekly
- **Evaluation** mainly from projects (individual – 4 to 5 per course): 70% to 80% of a student's final grade
- Original project (topic of their choice; possibly in pairs) as summative evaluation

Institutional decision since 2001

- **Programming language**: Java, Maple, VB.net, C++, Python
- **Mathematics Content**: Instructor's choice

Buteau, Muller & Ralph (2015). In *Online Proceedings of Coding+Math Symposium*

Content Example in MICA I-II-III

Main Learning Objective: Learn to design, program, and use interactive environments in order to conduct pure or applied mathematical investigations

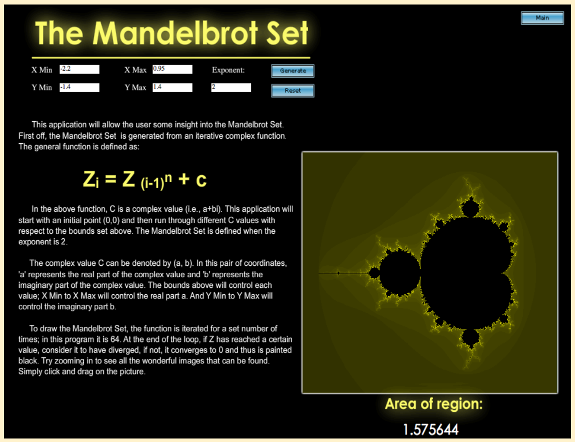
* Will need to be revised:
- Since 2020: coding in Gr.1-8 math curriculum
- Since 2021: also in Gr. 9

Year	Project	Topic	
MICA I*	EO 1	Conjecture about primes or hailstone sequence	
	EO 2	RSA encryption method	
	EO 3	Discrete dynamical system (cubic with two parameters)	
	EO 4	**Original, end-of-term project**	
Cohorts		***(Ralph's 2011-12 cohort)***	***(Fuks' 2014-15 cohort)***
MICA II	EO 5	Buffon needle problem & Monte Carlo integration	Discrete equations: Model of water pollution in a system of two connected lakes connected by a stream
	EO 6	Stats application to stock market	Systems of discrete equations: models of the spread of infectious diseases
	EO 7	Synchronization of traffic lights	Dynamical system of the logistic function & bifurcation diagram
	EO 8	Markov chains applied to income demographics and chronic illness	Stochastic models of bacterial growth
	EO 9	**Original, end-of-term project**	
MICA III	EO 10	Dynamical system of the logistic function & bifurcation diagram	Cellular automata models of road traffic flow
	EO 11	Simulation of battles (Lanchester equations)	Empirical modes and curve fitting: investigations of the Zipf's law.
	EO 12	Prey-predator biological model (Lotka-Volterra)	ODE models in pharmacokinetics, Michaelis-Menten equation
	EO 13	Cellular automata, simulation of epidemics & costs	Pursuit problems in 2D, Hathaway's circular pursuit
	EO 14	**Original, end-of-term project**	

Buteau, Muller & Ralph (2015). In *Online Proceedings of Coding+Math Symposium*

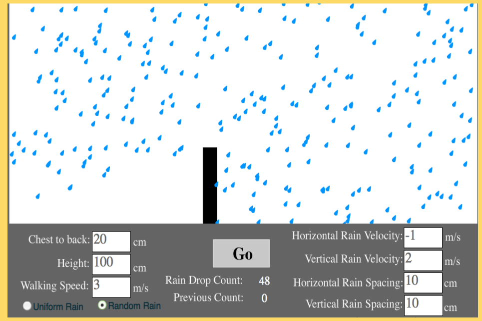
3. COURSE DESIGN AND CONTENT: STUDENT ORIGINAL FINAL PROJECTS

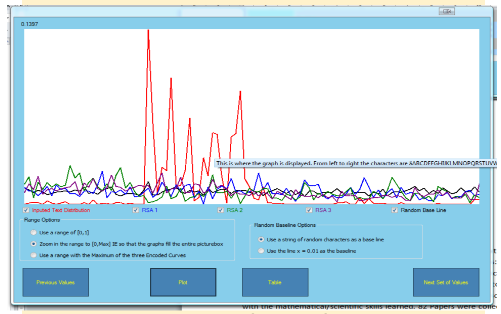
Examples of Students' Original Projects



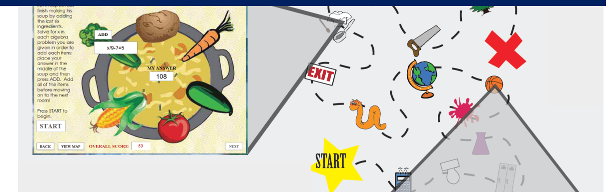
Exploration of the inner symmetries of the Mandelbrot set, and its **bounded area as the exponent** of the iterative complex function defining the Mandelbrot set **increases.** (A. Profetto; MICA II)

Is it better to walk or run in the rain? (K. Maheu & M. Lilly; **MICA II**)

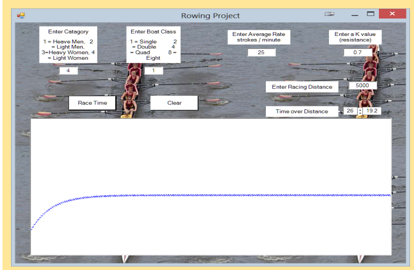




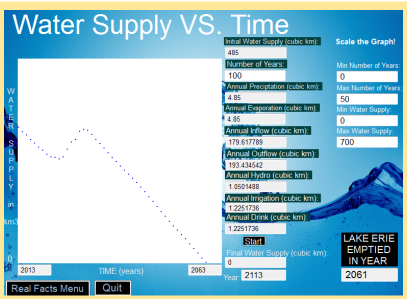
Encryption Pseudorandom-ness Explorer (N. Marshall; **MICA I**)



"Gearing Up for Grade 9" Learning Object, including tips, lessons, worksheets, and interactive games; tested with teachers and students (L. Broley & K. Wamboldt, **MICA I**)

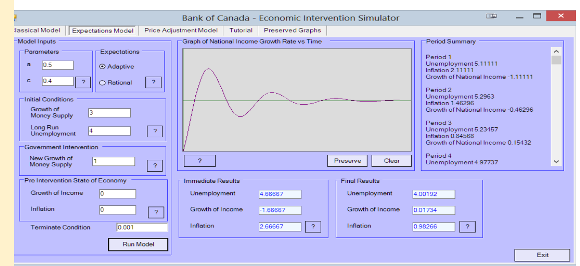


Simulation of rowing-boat running time (P. MacGregor; **MICA II**)



Exploration of the changes in the water supply of Lake Erie (Ontario) **over time** and explain why and how it changes (Ramona & M. Griffith, & A. Sones ; **MICA III**)

Simulation of the effects on the economy of central banks' monetary policy according to the Price Adjustment Model (G.Billard & R. Bruno; **MICA I**)



MICA Mathematics Integrated with Computers and Applications

'Guiding' Students

In Lectures:

"the relationship of the teacher to learner is very different: **the teacher introduces the learner to the microworld in which discoveries will be made, rather than to the discovery itself**"

(Papert, 1980, p. 209)

→ Part 1 of the student's engagement

Project Guidelines:

→ All parts of the student's engagement

In labs:

→ Mainly parts 2-3 of the student's engagement

"[Instructor] coming around to all the computer checking on everyone's work had a huge impact. [He/she] kind of forces you to ask questions which is really helpful because at times I don't like asking questions because I feel stupid but [he/she] doesn't make you feel stupid by any means." (MICA III student)

This assignment concerns an exploration, by use of graphic ... described by the Lotka-Volterra equations, namely:

$$\frac{dR}{dt} = kR - aRW$$

$$\frac{dW}{dt} = -rW + bRW$$

where a, b, k, and r are positive constants, time t is in months, and R (for 'rabbits') represents the prey population and W (for 'wolves') represents the predator population. As well of its modified version

…

PART A - Your program in vb.net

Modify Euler's method in order to solve numerically either of the two systems of equations by use of a 3-dimensional dynamical system involving t_n, R_n, and W_n, and display i) the numerical solution as graphs (in a same plot though with two different scaling for the vertical axis) of the pair R(t) and W(t), as well as ii) the phase trajectory for initial conditions given by a mouse click in the trajectory plane. The following gives more details about the programming of your interactive environment:

- Your interface will contain two picture boxes – give them descriptive names (e.g. PopulationGraphs and TrajectoryPlane).
- Your interface will also contain many textboxes for all the parameters involved in the model and in the graphical representations of the solutions. For example, a textbox for *MaxR*, the maximum

PART B – Your exploration using your interactive environment

Using your interactive environment, answer the following questions (and provide screen shots to support your arguments) about the model of aphids (prey) and ladybugs (predator) whereby we set k=2, a=0.01, r=0.5, and b=0.0001. We also set initial conditions to R_0=1000 and W_0 = 200. …

1. Plot the population of both species, and describe how they evolve over time.
2. Determine the equilibrium points, describe what it means in the context of the modeled situation, and categorize them based on an exploration of the trajectories close to the equilibrium points: stable, unstable, or neither (i.e. periodic motions around the point).
3. Describe how the populations change when k is increased by a small amount. Explain why you would expect the model to behave this way.

MICA I-III Project Guidelines: www.ctuniversitymath.ca/category/teaching-resources/

Buteau, Sacristán, & Muller (2019). *In Constructivist Foundations*.

Buteau, Muller et al. (in press). In *The Mathematics Teacher in the Digital Era (2nd Edn)* + Broley, Ablorh, Buteau, Mgombelo, & Muller (submitted). In *INDRUM 2022*

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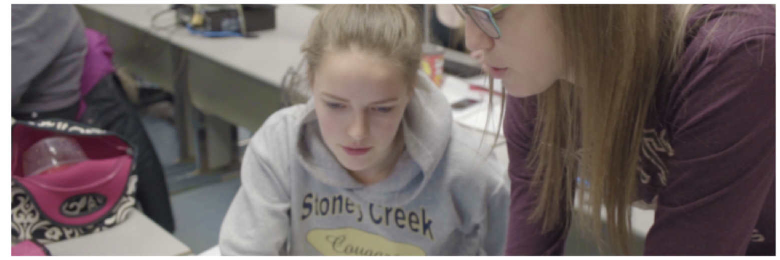
Teaching approach (guidance) for final projects



- The assignment projects completed during the course serve as models for students
- Lectures stop
- The instructor plays a mentor role, particularly for part I of the student engagement: *formulation of problem/conjecture, feasibility, added value of programming for the inquiry, and resources*
- Some instructors provide a list of suggested topics (basic ideas to build on)
- Teaching assistants mainly help with the programming of mathematics

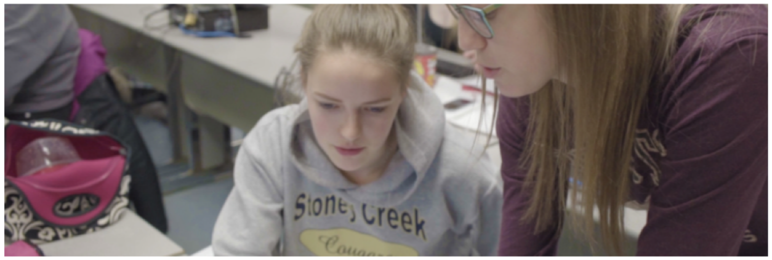
Buteau, Sacristán, & Muller (2019). *In Constructivist Foundations*.

Key teaching features, according to students

(anonymous questionnaire, MICA I-II-III, n=43)

- One-on-one **help with programming**, making example codes available, providing additional programming information to the class on demand,...
- A **class atmosphere** that promotes a safe environment to ask questions, allows for working individually or collaboratively, and individual contribution to the math content presentation.
- **Individual instructor/TA interventions** (e.g. in labs), such as re-explaining multiple times when needed, individually explaining what a student is doing wrong and why, guiding towards rather than telling the answer, etc.
- Impactful ways of being (e.g. **kind and supportive)** for **instructor/TA**
- **Organization of the course** enabling multiple modes of individual help (e.g. in labs)

Broley, Ablorh, Buteau, Mgombelo, & Muller (submitted). In *INDRUM*



What students find most challenging, and how they handle it

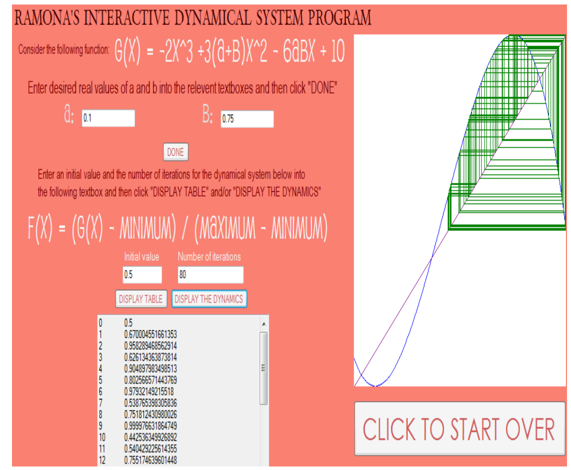
(anonymous questionnaire, MICA I-II-III, n=73)

- **What?** Programming (of the math)
- **How?** *Seeking help (62%) and/*or *Work on own (53%)* by, e.g.:
  - Independently gaining knowledge required to address the challenge
  - Taking their time and persevering
  - Preparing the math before programming it

Broley, Buteau, Levay, Marshall, Mgombelo, Muller, & Sardella (in press). In *RUME*

How student projects are evaluated

- Use of marking schemes
- Projects are evaluated on:
  - Design of interactive environment for the mathematical inquiry
  - Accuracy of the programmed mathematics
  - Programming norms (sub-procedures, comments, efficiency, etc.)
  - Report of their mathematical inquiry
  - Sometimes: ~10% on creativity, e.g. by extending the assigned inquiry
- Final project marking schemes have an additional component pertaining to the formulation of the problem/conjecture (originality, depth,…)



ALSO, a MICA course contains:

- 2 math standard math tests (~20%)
- Programming quizzes in MICA I (~9%)
- Sometimes: oral presentation of their final projects (~5%)

Buteau & Muller (2016). In *DEME*

For marking scheme examples, see the MICA I-III Project Guidelines: www.ctuniversitymath.ca/category/teaching-resources/

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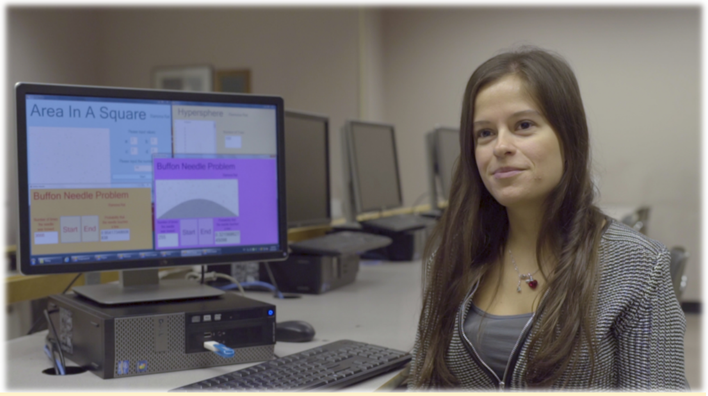
Students' Views



"**The main standout** between the assignments given in the MICA courses and those given in other math courses is the **ability to have the related concepts permanently embedded in your brain**. I believe this embedding occurs **because we write codes to solve the problems from scratch**. If I had not built the programs on my own, I would not understand the way that they work.
(Ramona, MICA III)

Buteau, Muller, Marshall, Sacristán, & Mgombelo (2016). In *DEME*

« The courses teach you how to use an interactive programming environment [...] and allows you to use it to investigate different mathematical theorems and concepts. **It is very effective because it allows you to make your own program to be able to see how this concept works, and play around with it to reach a further understanding of the concept.** »
(a MICA III student)

Buteau, Muller, & Marshall (2015). *In DEME*

Almost **50% of students** in MICA I-II-III who filled an anonymous online questionnaire (n=43) indicated **the final project** as being the **one from which they learned the most**.

Broley, Ablorh, et al. (in press). In *RUME*

Implications: A PBL approach may be effective to integrate programming in university math

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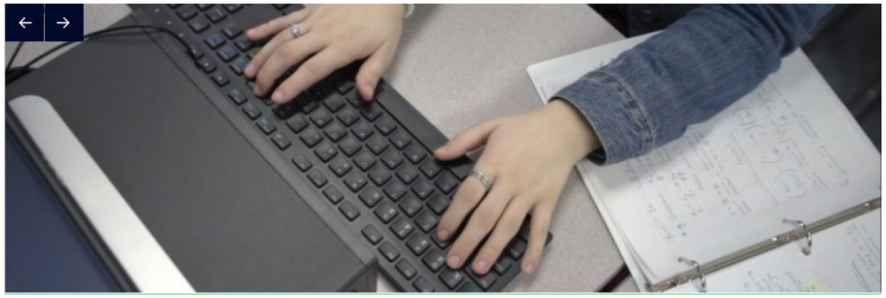
Constraints and Conditions for such an Inquiry Approach

- The course curriculum requires « white space »; i.e., one should consider:
  - →avoiding to overload the course content
  - →planning for significant classroom time dedicated to students engaging in inquiry projects (e.g., the 2-hour weekly labs in the MICA courses).
- One of the projects is original (required constraint)
- From our experience, having kowledgeable teaching assistants is essential for supporting the work of the instructor in courses such as MICA.
- Such a PBL approach (to integrating programming in mathematics courses) works well particularly with small cohorts
- Consensus among instructors concerning programming languages may be preferrable (if there is a sequence of courses)
- The course content can be flexible to the instructor's interests. The content closely aligns with the instructor's selection of projects.



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Mathematics Integrated with Computers and Applications

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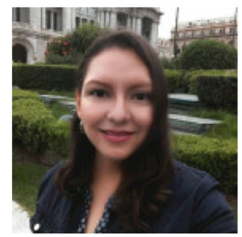
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SELECTED REFERENCES

THANK YOU!

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Slides at ctuniversitymath.ca/category/results-and-recommendations/

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